ESTIMATES OF GEODESIC RESTRICTIONS OF EIGENFUNCTIONS ON HYPERBOLIC SURFACES AND REPRESENTATION THEORY

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ABSTRACT. We consider restrictions along closed geodesics and geodesic circles for eigenfunctions of the Laplace-Beltrami operator on a compact hyperbolic Riemann surface. We obtain bounds on the L^2-norm and on the generalized periods of such restrictions as the corresponding eigenvalue tends to infinity. We use methods from the theory of automorphic functions and in particular the uniqueness of the corresponding invariant functionals on irreducible unitary representations of $PGL_2(\mathbb{R})$.

1. INTRODUCTION

1.1. **Maass forms.** Let Y be a compact Riemann surface with a Riemannian metric of constant curvature -1. We denote by dv the associated volume element and by $d(\cdot, \cdot)$ the corresponding distance function. The corresponding Laplace-Beltrami operator Δ is non-negative and has purely discrete spectrum on the space $L^2(Y, dv)$ of functions on Y. We will denote by $0 = \mu_0 < \mu_1 \leq \mu_2 \leq ...$ the eigenvalues of Δ and by ϕ_i the corresponding eigenfunctions (normalized to have L^2 norm one). In the theory of automorphic functions the functions ϕ_i are called non-holomorphic forms, Maass forms (after H. Maass, [M]) or simply automorphic functions.

The study of Maass forms is important in analysis, number theory and in many areas of mathematics and of mathematical physics. In particular, various questions concerning analytic properties of the eigenfunctions ϕ_i drew a lot of attention in recent years (see surveys [Sa1], [Sa2] and references therein).

1.2. **Restrictions to curves.** In this paper we study analytic properties of restrictions of eigenfunctions to closed curves.

Let $\gamma \subset Y$ be a smooth closed curve with the corresponding line element $d\gamma$. We fix a parametrization $t_\gamma : S^1 \to \gamma$ which we assume is proportional to the natural parametrization (i.e. $d\gamma^* = length(\gamma)d\theta$, where $0 \leq \theta < 1$ is the parametrization of S^1). Our main object of the study is the restriction $\phi_i^\gamma(\theta) = \phi_i(t_\gamma(\theta))$ of a Maass form ϕ_i to the curve γ.

The first question naturally arising in this setting asks for a bound on the size of the L^2-norm of the restriction of ϕ_i to γ:

$$(1) \qquad\qquad p^\gamma(\phi_i) = \int_\gamma |\phi_i|^2 d\gamma$$

as the eigenvalue $\mu_i \to \infty$.

We study the L^2-norm of such a restriction via generalized or twisted periods of a Maass form along the closed parameterized curve. Such periods are naturally arise in the theory of automorphic functions and are of interest in their own right. Namely, as a function on S^1, the restriction ϕ_i^γ gives rise to the following Fourier coefficients :

$$(2) \qquad\qquad p_n^\gamma(\phi_i) = \int_\gamma \phi_i(t_\gamma(\theta))e^{-2\pi in\theta} d\gamma \ .$$

We consider two types of curves on the Riemann surface Y: closed geodesics and geodesic circles. It turns out that for these special curves one can study the corresponding Fourier coefficients of Maass forms via representation theory.

1.2.1. *Geodesic circles.* We consider geodesic circles first. Let $\sigma = \sigma(r, y) \subset Y$ be a geodesic circle of a radius $r > 0$ centered at $y \in Y$. For r which is less than the injectivity radius of Y at y the geodesic circle is defined by $\sigma(r, y) = \{y' \in Y | \ \mathrm{d}(y, y') = r\}$. Writing the Laplace-Beltrami operator in polar geodesic coordinates centered at y and using the separation of variables one sees immediately that there exists a function $C_\mu(r, n)$ such that the n'th generalized period of a Maass form ϕ_i with the eigenvalue μ_i along $\sigma(r, y)$ is equal to

$$(3) \qquad\qquad p_n^{\sigma(r,y)}(\phi_i) = \int_{\sigma(r,y)} \phi_i(t_\sigma(\theta))e^{-2\pi in\theta} d\sigma = a_n^\sigma(\phi_i)C_{\mu_i}(r, n).$$

In other words, the restriction of the Maass form ϕ_i to the geodesic circle $\sigma(r, y)$ has the Fourier series expansion given by

$$(4) \qquad\qquad \phi_i(t_\sigma(\theta)) = \sum_n a_n^\sigma(\phi_i)C_{\mu_i}(r, n)e^{2\pi in\theta} \ , \ n \in \mathbb{Z}.$$

We stress that the function $C_\mu(r, n)$ depends only on the eigenvalue of ϕ_i and not on the choice of the eigenfunction (it is essentially equal to the appropriate hypergeometric function or Legendre function, see [He], [Sa3], [Pe]) and is independent of the point y. On the other hand coefficients $a_n^\sigma(\phi_i, y)$ capture the structure of the eigenfunction ϕ_i. The expansion (4) is similar to the Taylor expansion at the point $y \in Y$, and in fact would be the Taylor expansion for the holomorphic forms.

Coefficients $a_n^\sigma(\phi_i, y)$ are the main object of our study. Our main result is the following

Theorem A. *For any fixed geodesic circle $\sigma = \sigma(r, y) \subset Y$ there exists a constant C_σ such that for any eigenfunction ϕ_i with the eigenvalue μ_i the following bound holds*

$$(5) \qquad \sum_{|n| \leq T} |a_n^\sigma(\phi_i)|^2 \leq C_\sigma \cdot \max\{T, \sqrt{\mu_i}\} \ ,$$

for any $T \geq 1$.

In fact, we will prove this theorem for σ being any *fixed* image (not necessary smooth) of a circle in a tangent space $T_y Y$ at y under the exponential map $\exp_y : T_y Y \to Y$.

As a corollary we obtain the following bound on the L^2-norm of the restriction.

Corollary A. *Under the same conditions as in the Theorem A there exists a constant C'_σ such that the following bound holds*

$$(6) \qquad p^\sigma(\phi_i) \leq C'_\sigma \cdot \mu_i^{\frac{1}{6}} \ .$$

It is natural to expect that the true value of the norm for the restriction is given by the following

Conjecture A. *Let $\sigma \subset Y$ be a fixed closed geodesic circle then*

$$(7) \qquad p^\sigma(\phi_i) \ll \mu_i^\varepsilon$$

for any $\varepsilon > 0$.

We also obtain a uniform bound for the period.

Corollary A'. *Under the same conditions as in the Theorem A there exists a constant C''_σ such that the following bound holds*

$$(8) \qquad |p_0^{\sigma(r,y)}(\phi_i)| = \left| \int_{\sigma(r,y)} \phi_i(t_\sigma(\theta)) d\sigma \right| \leq C''_\sigma \ .$$

In fact, one can formulate a similar bound for the generalized period $p_0^{\sigma(r,y)}(\phi_i)$ for any n (see Section 5). On the basis of the Lindelof conjecture, one expects that the bound $|\int_{\sigma(r,y)} \phi_i(t_\sigma(\theta)) d\sigma| \ll \mu_i^{-1/4+\varepsilon}$ holds, but no improvement over (13) is known for general surfaces. For arithmetic surfaces and special circles (coming from imaginary quadratic fields) non-trivial improvements follow from the subconvexity bounds for the corresponding L-functions (see [MV]). We note that the analogous to (13) bound is sharp on the sphere S^2 and on the torus T^2.

1.2.2. *Closed geodesics.* We have a similar statement for closed geodesics as well. Namely, there exists a function $G_\mu(l, n)$ such that for any closed geodesic ℓ of a length l the corresponding period is given by

$$(9) \qquad p_n^\ell(\phi_{\mu_i}) = \int_\ell \phi_i(t_\ell(\theta)) e^{-2\pi i n\theta} d\ell = a_n^\ell(\phi_i) G_{\mu_i}(l, n).$$

The existence of the function $G_\mu(l, n)$ again follows from the separation of variables in coordinates corresponding to the first coordinate being the natural parameter along the geodesic ℓ and the second coordinate being the distance to the geodesic ℓ. It is also can be described in terms of an appropriate hypergeometric function. In Section 4 we will show how the existence of the function $G_\mu(l, n)$ easily follows from the representation theory.

Theorem B. *For any closed geodesic $\ell \subset Y$ there exists a constant C_ℓ such that for any eigenfunction ϕ_i with the eigenvalue μ_i the following bound holds*

$$\tag{10} \sum_{|n| \leq T} |a_n^\ell(\phi_i)|^2 \leq C_\ell \cdot \max\{T, \sqrt{\mu_i}\} \ ,$$

for any $T \geq 1$.

We have the similar

Corollary B. *Under the same conditions as in the Theorem B there exists a constant C_ℓ' such that the following bound holds*

$$\tag{11} p^\ell(\phi_i) \leq C_\ell' \cdot \mu_i^{\frac{1}{4}} \ .$$

The reason for the difference in exponents in Corollaries A and B is explained in Sections 4 and 5.

It is natural to expect that the true value of the norm for the restriction is given by the following

Conjecture B. *Let $\ell \subset Y$ be a fixed closed geodesic then*

$$\tag{12} p^\ell(\phi_i) \ll \mu_i^\varepsilon$$

for any $\varepsilon > 0$.

We also obtain a uniform bound for the period (compare to [Pitt]).

Corollary B'. *Under the same conditions as in the Theorem A there exists a constant C_ℓ'' such that the following bound holds*

$$\tag{13} |p_0^\ell(\phi_i)| = \left| \int_\ell \phi_i(t_\ell(\theta)) d\sigma \right| \leq C_\ell'' \ .$$

In fact, one can formulate a similar bound for the generalized period $p_0^\ell(\phi_i)$ for any n (see Section 4). On the basis of the Lindelof conjecture, one expects that the bound $| \int_\ell \phi_i(t_\sigma(\theta)) d\sigma | \ll \mu_i^{-1/4+\varepsilon}$ holds, but no improvement over (13) is known for general surfaces. For arithmetic surfaces and special circles (coming from imaginary quadratic fields) non-trivial improvements follow from the subconvexity bounds for the corresponding L-functions (see [MV]). We note that the analogous to (13) bound is sharp on the sphere S^2 and on the torus T^2.

1.3. **The method.** We follow the general strategy formulated in [BR3]. It is based on ideas from the representation theory of the group $PGL_2(\mathbb{R})$. Namely, any Riemannian surface Y gives rise to a discrete subgroup $\Gamma \subset PGL_2(\mathbb{R}) = G$ and the quotient space $X = \Gamma \backslash G$. We first use the fact that every Maass form ϕ generates an irreducible unitary representation $V \subset L^2(X)$ of G, called an automorphic representation. The eigenfunction ϕ corresponds to a unit vector $e_0 \in V$ invariant under the compact subgroup $K \subset G$ such that $X/K \simeq Y$ (such a vector is unique up to multiplication by a constant). All irreducible unitary representations of G are classified and have *explicit models*. This setting, formulated by I. Gel'fand and S. Fomin ([GF]), will be essential in what follows.

Generalized periods along closed geodesics and geodesic circles are related to the representation theory in the following way. Any closed geodesic ℓ gives rise to a closed orbit $\mathcal{O}_\ell$ of the diagonal subgroup $A \subset G$ under the right action of G on X and similarly each geodesic circle σ gives rise to an orbit $\mathcal{O}_\sigma$ of an appropriate compact subgroup K' of G. We note that any closed orbit $\mathcal{O}$ of A gives rise to the cyclic subgroup $A_\mathcal{O} \subset A$ of elements in A acting trivially on $\mathcal{O}$. The quotient group $A/A_\mathcal{O}$ is compact (this is also true for an orbit of a compact subgroup but we will not use this subgroup since K' is compact by itself). The L^2-form on the orbit $\mathcal{O}$ gives rise to a non-negative Hermitian form $H_\mathcal{O}$ on the space of smooth functions on X via restriction. We study the coefficients $p^{\ell/\sigma}(\phi_i)$ and $p_n^{\ell/\sigma}(\phi_i)$ through this Hermitian form.

Let $\mathcal{G} \subset G$ stands for either A or K'. For an orbit $\mathcal{O}$ of $\mathcal{G}$ as above we study the form $H_\mathcal{O}$ by means of the corresponding generalized periods along $\mathcal{O}$. Namely, for a unitary character $\chi : \mathcal{G} \to \mathbb{C}$ we consider a functional on the space $V \cap C^\infty(X)$, which we call the χ-period along the orbit $\mathcal{O}$, defined on $V \subset C^\infty(X)$ by via the integral against $\bar{\chi}$ over $\mathcal{O}$ with respect to a $\mathcal{G}$-invariant measure. For $\mathcal{G} = A$ we consider only characters trivial on $A_\mathcal{O}$. One have the Plancherel formula expressing the value of the form $H_\mathcal{O}$ on a vector in V as the sum of squares of the coefficients arising from χ-periods of the same vector (see (17)). This corresponds to the usual Plancherel formula for the Fourier expansion of the restriction of a function to the orbit $\mathcal{O}$ under the identification $\mathcal{O} \simeq S^1$. On the other hand, any χ-period restricted to an automorphic representation V gives rise to a functional d_χ^{aut} which is χ-equivariant under the action of $\mathcal{G}$ on V. As well-known in the representation theory of $PGL_2(\mathbb{R})$ the space of such functionals on a unitary irreducible representation of G is one-dimensional. We use this fact in order to introduce the coefficients $a_\chi \in \mathbb{C}$ defined below, which are the main object of our study. Namely, using an *explicit model* of an irreducible automorphic representation in the space of functions on the line (for $\mathcal{G} = A$) or on the circle (for $\mathcal{G} = K'$) we define a $(\chi, \mathcal{G})$-equivariant functional d_χ^{mod} by means of an *explicit kernel* (see (22)). The uniqueness of such functionals implies the existence of the coefficient of proportionality

$$d_\chi^{aut} = a_\chi \cdot d_\chi^{mod} \ .$$

The main idea of our method is to use analytic properties of the explicitly constructed functionals d_χ^{mod} in order to control the coefficients in the Plancherel decomposition of $H_\mathcal{O}$. The study of the functionals d_χ^{mod} is based on the stationary phase method. We

use simple geometric properties of the form $H_{\mathcal{O}}$ (see Lemmas 3.2.1 and 5.2) in order to obtain sharp on the average bound on the coefficients a_χ (Theorems 3.4 and 5.2.2) and use it to prove Theorems A and B. The discrepancy in the exponent in Corollaries A and B is a reflection of different spectral decomposition of the same K-fixed vector $e_0 \in V$ with respect to characters of A and of K' respectively (compare (31) to (43); see Remark 5.3.1).

1.4. **Remarks.** 1. The supremum norm of an eigenfunction ϕ_μ of the Laplace-Beltrami operator on a compact Riemannian manifold M, $\dim M = n$, satisfies the Hörmander's classical bound $\sup |\phi_\mu| \leq c\mu^{\frac{n-1}{4}} \|\phi_\mu\|_{L^2(M)}$, where μ is the eigenvalue of ϕ_μ. This bound is sharp on the standard sphere. Hence bounds in Corollaries A and B do not follow from the general pointwise bound on ϕ_i.

After announcing the proof of bounds in Corollaries A and B author have learned that these results are special cases of results of D. Tataru ([Ta]) and also follow from recent results of N. Burq, P. Gérard and N. Tzvetkov ([BGT1]). In fact, Tataru showed that the estimate $p_\gamma(\phi_i) \leq C_\gamma \mu_i^{1/6}$ holds for any smooth non-flat curve $\gamma \subset Y$ and $p^\gamma(\phi_i) \leq C_\gamma \mu_i^{\frac{1}{4}}$ for a flat smooth curve. For general Y the second bound is sharp. Namely, one can see that on the standard sphere for each eigenvalue $\mu = n(n+1)$ and for each geodesic ℓ (i.e. an equator) there exists an eigenfunction $Y_{\mu,\ell}$ (e.g. the lowest associated spherical harmonic Y_n^n, see [Ma]) of the L^2-norm one such that $\int_\ell |Y_{\mu,\ell}|^2 d\ell$ is of order $\mu^{\frac{1}{4}}$.

Moreover, further results in [BGT2] greatly extended our results to L^p-norms and made the present paper obsolete. We also mention [Bo] and [So] among further developments which reversed [BGT2] and showed the realtion to L^4-norms of eigenfunctions on Y.

2. We conjecture that the bound $p^{\ell/\sigma}(\phi_i) \ll \mu^\varepsilon$ holds. This is consistent with the conjecture of P. Sarnak [Sa1] claiming that $\sup |\phi_i| \ll \mu^\varepsilon$. Unfortunately, for a general compact hyperbolic surface Y the only known improvement in the bound for the supremum of an eigenfunction is logarithmic: $\sup |\phi_i| \leq C\mu_i^{1/4} / \ln \mu_i$ ([Be]).

We also conjecture that bounds $|a_n^{\sigma/\ell}(\phi_i)| \ll \max\{n^\varepsilon, \mu_i^\varepsilon\}$ hold for any $\varepsilon > 0$.

For Riemann surfaces of the number-theoretic origins and the special basis of eigenfunctions of number-theoretic significance (the so-called Hecke-Maass basis) H. Iwaniec and P. Sarnak [ISa] have improved the exponent $1/4$ in the supremum norm bound above to the exponent $5/24$. Corolaries A and B show that the quantities $p^{\ell/\sigma}(\phi_i)$ are more accessible than the supremum of eigenfunctions.

We also note that for 3-dimensional hyperbolic manifolds the bound $\sup |\phi_i| \ll \mu^\varepsilon$ does not hold in general as was shown by Z. Rudnick and P. Sarnak [RS] and hence one can not expect that the corresponding generalized periods on geodesic spheres are small for all eigenfunctions. It is interesting to study corresponding periods along closed geodesics in order to see if these are small with respect to the eigenvalue.

3. The generalized periods of the restriction of an eigenfunction along a closed geodesic and along a geodesic circle are of utmost interest in number theory (see [Du], [KSa]). These are the coefficients a_χ defined in 1.3 (see (22) and (39) for the exact definition). Our main underlying results (Theorems A and B) give sharp mean value bound on these coefficients. These results are similar to the bound of G. Hardy on the average size of Fourier coefficients of cusp forms. We formulate a conjecture concerning the size of these periods (Conjecture 3.4.3) which should be viewed as an analog of the Ramanujan-Petersson conjecture on Fourier coefficients of cusp forms. In the special case of Hecke–Maass forms periods along closed geodesics and values at some special points (i.e. Heegner points) give rise to special values of L-functions (see [KSa] and references therein). In these cases using so-called convexity bounds on these L-functions one obtains *stronger* bounds on periods along these special curves then the bounds which trivially follow from Theorems A and B. It is an intriguing question if there exists a connection between generalized periods and special values of L-functions in general.

4. In this paper we treat restrictions to a *fixed* closed geodesic. It is a deep problem to understand the dependence of the norm of these restrictions and the corresponding periods on the geodesic (e.g. to study the dependence of the constant C_ℓ on the geodesic ℓ).

The similar question for geodesic circles is of utmost interest in the spectral theory. In particular, one would like to study the behavior of $p_\sigma(\phi_i)$ for *small* geodesic circles with the radius of order $\mu_i^{-\alpha}$ for $\frac{1}{2} \leq \alpha < 0$. This is related to so-called doubling constant of ϕ_i (see [NPS]).

5. Our methods work for non-compact hyperbolic surfaces of finite volume as well. In fact, for non-compact Riemann surfaces of finite volume one can pose a similar question for the size of restrictions to a horocycle instead of a geodesic. Namely, let $\eta \in Y$ be a fixed horocycle (there is a continuous family of horocycles associated to each cusp of Y). Similarly to $p^\ell(\phi_i)$, one defines $p^\eta(\phi_i)$ as the L^2-norm of the corresponding restriction of the eigenfunction ϕ_i to η and the periods $p_n^\eta(\phi_i)$. Under the appropriate normalization the coefficients $p_n^\eta(\phi_i)$ are equal to the usual Fourier coefficients of Maass forms. One can show that for a cusp form ϕ_i the bound $p^\eta(\phi_i) \leq C_\eta \mu_i^{\frac{1}{6}}$ holds. Moreover, using nontrivial bounds on Fourier coefficients of cusp forms for Y arising from congruence subgroups one can show that for the Hecke-Maass basis of eigenfunctions on such surfaces the sharp bound $p^\eta(\phi_i) \leq C_\eta \mu_i^\varepsilon$ holds for any $\varepsilon > 0$. One expects that the similar bound holds for general Y. In order to improve the exponent $1/6$ above one have to improve over the subconvexity bound for Fourier coefficients of Maass forms obtained in [BR1]. The analog of Theorems A and B for Fourier coefficients of Maass forms was proved by A. Good [Go] following Hardy's method.

The paper is organized as follows. In Section 2 we remind the notion of an automorphic representation and the correspondence between eigenfunctions and automorphic representations. In Section 3 we introduce non-negative Hermitian forms associated with each

closed geodesic and prove basic inequalities for these forms. We also prove our main technical result, Theorem 3.4, and formulate Conjecture 3.4.3 on the size of the corresponding periods. In Section 4 we compute spectral density of a K-fixed vector and apply this to prove Theorems A. In Section 5 we prove the bound in Theorem B for restrictions to geodesic circles along similar lines.

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2. REPRESENTATION THEORY AND EIGENFUNCTIONS

It has been understood since the seminal works of A. Selberg [Se] and I. Gel'fand, S.Fomin [GF] that representation theory plays an important role in the study of eigenfunctions ϕ_i. Central for this role is the correspondence between eigenfunctions of Laplacian on Y and unitary irreducible representations of the group $PGL_2(\mathbb{R})$ (or what is more customary of $PSL_2(\mathbb{R})$). This correspondence allows one, quite often, to obtain results that are more refined than similar results for the general case of a Riemannian metric of variable curvature.

We remind the basic setting for the theory of automorphic functions (see the excellent source [G6] for the representation-theoretic point of view we adopt here and [Iw] for a more classical approach based on harmonic analysis on the upper half plane).

2.1. **Automorphic representations.** We start with the geometric construction which allows one to pass from analysis on a Riemann surface to representation theory.

One stars with with the upper half plane $\mathbb{H}$ equipped with the hyperbolic metric of constant curvature -1 (or equivalently one might work with a more "homogeneous" model of the Poincaré unit disk D; the use of $\mathbb{H}$ is more customary in the theory of automorphic functions). The group $SL_2(\mathbb{R})$ acts on $\mathbb{H}$ by the standard fractional linear transformations. This action allows one to identify the group $PSL_2(\mathbb{R})$ with the group of all orientation preserving motions of $\mathbb{H}$. For reasons explained bellow we would like to work with the group G of all motions of $\mathbb{H}$; this group is isomorphic to $PGL_2(\mathbb{R})$. Hence throughout the paper we denote $G = PGL_2(\mathbb{R})$.

Let us fix a discrete co-compact subgroup $\Gamma \subset G$ and set $Y = \Gamma \setminus \mathbb{H}$. We consider the Laplace operator on the Riemann surface Y and denote by μ_i its eigenvalues and by ϕ_i the corresponding normalized eigenfunctions.

The case when Γ acts freely on $\mathbb{H}$ precisely corresponds to the case discussed in 1.1 (this follows from the uniformization theorem for the Riemann surface Y). Our results hold for general co-compact subgroup Γ (and in fact for any lattice $\Gamma \subset G$).

We will identify the upper half plane $\mathbb{H}$ with G/K, where $K = PO(2)$ is a maximal compact subgroup of G (this follows from the fact that G acts transitively on $\mathbb{H}$ and the stabilizer in G of the point $z_0 = i \in \mathbb{H}$ coincides with K).

We denote by X the compact quotient $\Gamma \setminus G$ (we call it the automorphic space). In the case when Γ acts freely on $\mathbb{H}$ one can identify the space X with the bundle $S(Y)$ of unit tangent vectors to the Riemann surface $Y = \Gamma \setminus \mathbb{H}$.

The group G acts on X (from the right) and hence on the space of functions on X. We fix the unique G-invariant measure μ_X on X of total mass one. Let $L^2(X) = L^2(X, d\mu_X)$ be the space of square integrable functions and $(\Pi_X, G, L^2(X))$ the corresponding unitary representation. We will denote by P_X the Hermitian form on $L^2(X)$ given by the scalar product. We denote by $||\ ||_X$ or simply $||\ ||$ the corresponding norm and by $\langle f, g \rangle_X$ the corresponding scalar product.

The identification $Y = \Gamma \setminus \mathbb{H} \simeq X/K$ induces the embedding $L^2(Y) \subset L^2(X)$. We will always identify the space $L^2(Y)$ with the subspace of K-invariant functions in $L^2(X)$.

Let ϕ be a normalized eigenfunction of the Laplace-Beltrami operator on Y. Consider the closed G-invariant subspace $L_\phi \subset L^2(X)$ generated by ϕ under the action of G. It is well-known that $(\pi, L) = (\pi_\phi, L_\phi)$ is an irreducible unitary representation of G (see [G6]).

Usually it is more convenient to work with the space $V = L^\infty$ of smooth vectors in L. The unitary Hermitian form P_X on V is G-invariant.

A smooth representation (π, G, V) equipped with a positive G-invariant Hermitian form P we will call a *smooth pre-unitary representation*; this simply means that V is the space of smooth vectors in the unitary representation obtained from V by completion with respect to P.

Thus starting with an automorphic function ϕ we constructed an irreducible smooth pre-unitary representation (π, V). In fact we constructed this space together with a canonical morphism $\nu : V \to C^\infty(X)$ since $C^\infty(X)$ is the smooth part of $L^2(X)$.

Definition. A smooth pre-unitary representation (π, G, V) equipped with a G-morphism $\nu : V \to C^\infty(X)$ we will call an X-*automorphic representation*.

We will assume that the morphism ν is normalized, i.e. it carries the standard L^2 Hermitian form P_X on $C^\infty(X)$ into Hermitian form P on V.

Thus starting with an automorphic function ϕ we constructed

(i) an X-automorphic irreducible pre-unitary representation (π, V, ν),

(ii) a K-invariant unit vector $e_V \in V$ (this vector is just our function ϕ).

Conversely, suppose we are given an irreducible smooth pre-unitary X-automorphic representation (π, V, ν) of the group G and a K-fixed unit vector $e_V \in V$. Then the function $\phi = \nu(e_V) \in C^\infty(X)$ is K-invariant and hence can be considered as a function on Y. The fact that the representation (π, V) is irreducible implies that ϕ is an automorphic function, i.e. an eigenfunction of Laplacian on Y.

Thus we have established a natural correspondence between Maass forms ϕ and tuples (π, V, ν, e_V), where (π, V, ν) is an X-automorphic irreducible smooth pre-unitary representation and $e_V \in V$ is a unit K-invariant vector.

It is well known that for X compact the representation $(\Pi_X, G, L^2(X))$ decomposes into a direct (infinite) sum

$$(14) \qquad\qquad\qquad L^2(X) = \oplus_j (\pi_j, L_j)$$

of irreducible unitary representations of G (all representations appear with finite multiplicities (see [G6])). Let (π, L) be one of these irreducible "automorphic" representations and $V = L^\infty$ its smooth part. By definition V is given with a G-automorphic isometric morphism $\nu : V \to C^\infty(X)$, i.e. V is an X-automorphic representation.

If V has a K-invariant vector it corresponds to a Maass form. There are other spaces in this decomposition which correspond to discrete series representations. Since they are not related to Maass forms we will not study them in more detail.

2.2. **Representations of** $PGL_2(\mathbb{R})$. All irreducible unitary representations of G are classified. For simplicity we consider only those with a nonzero K-fixed vector (so called representations of class one) since only these representations arise from Maass forms. These are the representations of the principal and the complementary series and the trivial representation.

We will use the following standard explicit model for irreducible smooth representations of G.

For every complex number λ consider the space V_λ of smooth even homogeneous functions on $\mathbb{R}^2 \setminus 0$ of homogeneous degree $\lambda - 1$ (which means that $f(ax, ay) = |a|^{\lambda-1} f(x, y)$ for all $a \in \mathbb{R} \setminus 0$). The representation (π_λ, V_λ) is induced by the action of the group $GL_2(\mathbb{R})$ given by $\pi_\lambda(g) f(x, y) = f(g^{-1}(x, y)) |\det g|^{(\lambda-1)/2}$. This action is trivial on the center of $GL_2(\mathbb{R})$ and hence defines a representation of G. The representation (π_λ, V_λ) is called *representation of the generalized principal series*.

When $\lambda = it$ is purely imaginary the representation (π_λ, V_λ) is pre-unitary; the G-invariant scalar product in V_λ is given by $\langle f, g \rangle_{\pi_\lambda} = \frac{1}{2\pi} \int_{S^1} f \bar{g} d\theta$. These representations are called representations of *the principal series*.

When $\lambda \in (-1, 1)$ the representation (π_λ, V_λ) is called a representation of the complementary series. These representations are also pre-unitary, but the formula for the scalar product is more complicated (see [G5]). We will not discuss these since there are only finitely many such representations for each Γ and we are interested in properties of eigenfunctions ϕ_i as $\mu_i \to \infty$.

All these representations have K-invariant vectors. We fix a K-invariant unit vector $e_\lambda \in V_\lambda$ to be a function which is one on the unit circle in $\mathbb{R}^2$.

Representations of the principal and the complimentary series exhaust all nontrivial irreducible pre-unitary representations of G of class one ([G5], [L]).

Suppose we are given a class one X-automorphic representation $\nu : V_\lambda \to C^\infty(X)$; we assume ν to be an isometric embedding. Such ν gives rise to an eigenfunction of the Laplacian on the Riemann surface $Y = X/K$ as before. Namely, if $e_\lambda \in V_\lambda$ is a unit K-fixed vector then the function $\phi = \nu(e_\lambda)$ is a normalized eigenfunction of the Laplacian on the space $Y = X/K$ with the eigenvalue $\mu = \frac{1-\lambda^2}{4}$. This explains why λ is a natural parameter to describe Maass forms.

3. Closed geodesics, restrictions and periods

3.1. **Closed geodesics.** We use the well-known description of closed geodesics in terms of hyperbolic conjugacy classes in Γ (see [Iw]). The set of geodesics on $\mathbb{H}$ consist of semi-circles centered on the absolute ($Im(z) = 0$) and of vertical lines. From this and the presentation of Y as the quotient $\Gamma \setminus \mathbb{H}$ it follows that for each closed geodesic $\ell \subset Y$ there exist an associated hyperbolic element $\gamma \in \Gamma$, defined up to the conjugacy in Γ, such that γ stabilizes an appropriate geodesic $\widetilde{\ell} \subset \mathbb{H}$ and gives rise to the one-to-one projection $\{\gamma\} \setminus \widetilde{\ell} \to \ell$ under the map $\mathbb{H} \to \Gamma \setminus \mathbb{H} \simeq Y$. Let γ be such an element. We denote $\Gamma_\ell \subset \Gamma$ the cyclic subgroup generated by γ. The subgroup Γ_ℓ is defined up to the conjugacy in Γ. Under the described above correspondence simple closed geodesics correspond to conjugacy classes of *primitive* cyclic hyperbolic subgroups which are generated by primitive hyperbolic elements (i.e. those $\gamma' \in \Gamma$ satisfying $\gamma' \neq \gamma^n$ for any $\gamma \in \Gamma$).

We use the well-known reformulation of the above description of closed geodesics in Y in terms of closed orbits of the group of diagonal matrices $A \subset G$ acting on X on the right. Let Γ_ℓ be as above and let $\gamma \in \Gamma_\ell$ be its generator. Since $\gamma \in G$ is a hyperbolic element there exists an element $g_\gamma \in G$ such that $g_\gamma^{-1} \gamma g_\gamma = a_\gamma \in A$. We denote A_γ the subgroup generated by a_γ. The orbit $\mathcal{O}_\ell = g_\gamma \cdot A \subset X$ of A is a closed orbit which is homeomorphic to A/A_γ. Under the natural map $X \to Y$ the orbit $\mathcal{O}_\ell$ is mapped one-to-one onto the closed geodesic ℓ we start with. We denote by $d\mathcal{O}_\ell$ the unique A-invariant measure on $\mathcal{O}_\ell$ of the total mass one (a more geometric normalization which corresponds to the length of the geodesic ℓ would be $\int_{A/A_\gamma} 1 d^\times a$). We have hence the relation $p^\ell(\phi) = length(\ell) \int_{\mathcal{O}_\ell} |\phi|^2 d\mathcal{O}_\ell$.

3.2. **Restrictions and periods.** Let ℓ be a fixed closed geodesic and $\gamma \in \Gamma_\ell$, $\mathcal{O} = \mathcal{O}_\ell$ as above. Any such A-orbit $\mathcal{O}$ gives rise to natural Hermitian forms and a set of functionals on the space of functions $C^\infty(X)$. These will be our main tools in what follows.

We define $H_\mathcal{O}$ to be the non-negative Hermitian form on $C^\infty(X)$ given by

$$(15) \qquad\qquad H_\mathcal{O}(f,g) = \int_\mathcal{O} f(o)\bar{g}(o)d\mathcal{O}$$

for any $f, g \in C^\infty(X)$. We will use the shorthand notation $H_\mathcal{O}(f) = H_\mathcal{O}(f, f)$.

Let $\tilde{A}_\gamma$ be the set of characters $\chi : A \to S^1 \subset \mathbb{C}^\times$ trivial on A_γ. This is an infinite cyclic group generated by a character χ_1. Hence $\tilde{A}_\gamma = \{\chi_n = \chi_1^n, \ n \in \mathbb{Z}\}$. We fix a point $\dot{o} \in \mathcal{O}$. To a character $\chi \in \tilde{A}_\gamma$ we associate the function $\chi_. : \mathcal{O} \to S^1$ given by $\chi_.(\dot{o}a) = \chi(a)$. For a character $\chi \in \tilde{A}_\gamma$ we define the functional on $C^\infty(X)$ given by

$$(16) \qquad\qquad d^{aut}_{\chi,\mathcal{O}}(f) = \int_\mathcal{O} f(o)\bar{\chi}_.(o)d\mathcal{O}$$

for any $f \in C^\infty(X)$. The functional $d^{aut}_{\chi,\mathcal{O}}$ is χ-equivariant: $d^{aut}_{\chi,\mathcal{O}}(R(a)f) = \chi(a)d^{aut}_{\chi,\mathcal{O}}(f)$ for any $a \in A$, where R is the right action of G on the space of functions on X. For a given orbit $\mathcal{O}$ and a choice of a generator χ_1 we will use the shorthand notation $d^{aut}_n = d^{aut}_{\chi_n,\mathcal{O}}$.

The functions $\{(\chi_n)_.\}$ form an orthonormal basis for the space $L^2(\mathcal{O}, d\mathcal{O})$. Hence we have the standard Plancherel formula:

$$(17) \qquad\qquad H_\mathcal{O}(f) = \sum_n |d^{aut}_n(f)|^2 \ .$$

Let V be an automorphic representation. We consider the non-negative Hermitian form $H^V_\mathcal{O}$ on V given by the restriction of $H_\mathcal{O}$ and non-negative Hermitian forms of rank one $Q^{aut}_n(\cdot) = |d^{aut}_n(\cdot)|^2$ restricted to V. We rewrite (17) as

$$(18) \qquad\qquad H^V_\mathcal{O} = \sum_n Q^{aut}_n$$

and view this as an equality of non-negative Hermitian forms on V.

3.2.1. *Geometric inequality for $H_\mathcal{O}$.* The Hermitian form $H_\mathcal{O}$ is defined through the integral over a compact set in X and hence its average over the action of G is bounded by the standard Hermitian form P_X on $L^2(X)$. Namely, the group G naturally acts on the space of Hermitian forms on $C^\infty(X)$. We denote this action by Π. We extend Π to the action of the algebra $H(G) = C^\infty_c(G, \mathbb{R})$ of smooth real valued functions with compact support. We have the following basic

Lemma.   *For any $h \in H(G)$, $h \geq 0$ there exists a constant $C = C_h$ such that*

$$\Pi(h)H_\mathcal{O} \leq CP_X \ .$$

Proof. Let $u \in C^\infty(X)$. Then $P_X(u) = <\mu_X, |u|^2>$ and $\Pi(h)H_\mathcal{O}(u) = <\mu', |u|^2>$, where $\mu' = \Pi(h)d\mathcal{O}$. The measure μ' is smooth (since h is smooth) and X is compact hence the measure μ' is bounded by $C\mu_X$. $\qquad\square$

3.3. Homogeneous functionals.

3.3.1. *Uniqueness of homogeneous functionals.* The functionals d_n^{aut} introduced above are χ_n- equivariant. The central fact about such functionals is the following uniqueness result:

Theorem. *Let (π, V) be an irreducible smooth admissible representations of G and let $\chi : A \to \mathbb{C}^\times$ be a multiplicative character. Then $\dim \mathrm{Hom}_A(V, \chi) = 1$.*

The uniqueness statement is a standard fact in the representation theory of G. It easily follows from the existence of the Kirillov model (see [G6]). There is no uniqueness of trilinear functionals for representations of $SL_2(\mathbb{R})$ (the space is two-dimensional: it splits into even and odd functionals). This is the reason why we prefer to work with $PGL_2(\mathbb{R})$ (although our method could be easily adopted to $SL_2(\mathbb{R})$).

3.3.2. *Model homogeneous functionals.* For every $\lambda \in \mathbb{C}$ we denote by (π_λ, V_λ) the smooth class one representation of the generalized principle series of the group $G = PGL_2(\mathbb{R})$ described in 2.2. We will use the realization of (π_λ, V_λ) in the space of smooth homogeneous functions on $\mathbb{R}^2 \setminus 0$ of homogeneous degree $\lambda - 1$ (see [G5]).

For explicit computations it is often convenient to pass from plane model to a line model. Namely, the restriction of functions in V_λ to the line $(x, 1) \subset \mathbb{R}^2$ defines an isomorphism of the space V_λ with the space $C^\infty(\mathbb{R})$ of even smooth functions on $\mathbb{R}$ decaying on infinity as $|x|^{\lambda-1}$ so we can think about vectors in V_λ as functions on $\mathbb{R}$. Under such an identification the action of the diagonal subgroup is given by

$$(19) \qquad \pi_\lambda(a)f(x,1) = f(a^{-1}x, a) = |a|^{\lambda-1}f(a^{-2}x, 1) \ .$$

Where we used the shorthand notation $a = diag(a, a^{-1}) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$.

Note that in this model a K-fixed unit vector is given by $e_0^\lambda(x) = c(1 + x^2)^{(\lambda-1)/2}$ (where the normalization constant c is independent of λ).

Let $s \in \mathbb{C}$ and $\chi_s : A \to \mathbb{C}^\times$, $\chi_s(a) = |a|^s$ be the corresponding character. From the description of the action of A in the line model we see that the functionals (distributions) $d_{s,\lambda}^{mod}$ defined by the kernel $|x|^{-\frac{1}{2}-\lambda/2+s/2}$ are (A, χ_s)-equivariant functionals on V_λ. Namely, we define

$$(20) \qquad d_{s,\lambda}^{mod}(v) = \int |x|^{-\frac{1}{2}-\lambda/2+s/2}v(x)dx \ .$$

We have $d_{s,\lambda}^{mod}(\pi_\lambda(a)v) = \chi_s(a)d_{s,\lambda}^{mod}(v)$. In particular an A-invariant functional is given by

$$(21) \qquad\qquad d_{0,\lambda}^{mod}(v) = \int |x|^{-\frac{1}{2}-\lambda/2}v(x)dx .$$

We note that for the general value of $s \in \mathbb{C}$ one have to understand the above integrals in a regularized sense but we will be interested in unitary characters only ($s \in i\mathbb{R}$) for which the integrals above are absolutely convergent.

3.3.3. *Coefficients of proportionality.*

The uniqueness of homogeneous functionals implies that d^{aut} and d^{mod} are proportional. Namely, let $\mathcal{O}$ be a closed A-orbit, A_γ the corresponding subgroup with a generator $a_\gamma = diag(a_\gamma, a_\gamma^{-1})$ and let V be an automorphic representation isomorphic to a representation of the principal series V_λ. We denote by $q = q_\gamma = 1/\ln(a_\gamma)$. For $n \in \mathbb{Z}$ we consider the kernel $|x|^{-\frac{1}{2}-\lambda/2+inq}$ and the corresponding homogeneous functional $d_{n,\lambda}^{mod}$ defined on V_λ by means of this kernel. The set $\{d_{n,\lambda}^{mod}, \ n \in \mathbb{Z}\}$ exhaust the set of all χ-invariant functionals on V as $\chi \in \tilde{A}_\gamma$. It follows from the uniqueness theorem that for any $n \in \mathbb{Z}$ and any automorphic representation V which is isomorphic to V_λ there exists a constant $a_{n,V} \in \mathbb{C}$ such that

$$(22) \qquad\qquad d_{\chi_n,\mathcal{O}}^{aut} = a_{n,V} \cdot d_{n,\lambda}^{mod} .$$

The constant $a_{n,V}$ depends on the parameter λ and also on the isometry $\nu_V : V \to V_\lambda$ (e.g. when the multiplicity of the corresponding eigenvalue of Δ is greater than one). We will, however, use the notation $a_{n,\lambda} = a_{n,V}$ suppressing this difference as our method is not sensitive to the multiplicity of V.

We denote by $Q_{n,\lambda}^{mod}$ the non-negative Hermitian form $Q_{n,\lambda}^{mod}(\cdot) = |d_{n,\lambda}^{mod}(\cdot)|^2$. Taking into account (18) we arrive at our basic relation

$$(23) \qquad\qquad H_{\mathcal{O}}^V = \sum_n |a_{n,\lambda}|^2 Q_{n,\lambda}^{mod}$$

of non-negative Hermitian forms on V.

3.4. **Average bound.**

We formulate now our main result (Theorem B):

Theorem. *For a given orbit $\mathcal{O}$ as above there exists a constant $C = C_\mathcal{O}$ such that for any $T \geq 1$ the following bound holds*

$$(24) \qquad\qquad \sum_{|n|\leq T} |a_{n,\lambda}|^2 \leq C \cdot \max(T, |\lambda|) .$$

for any automorphic representation which is isomorphic to a representation of the principal series V_λ.

3.4.1. *Test functionals.* The proof of Theorem 3.4 is based on the notion of positive test functionals on the space $\mathcal{H}(V)$ of Hermitian forms on an automorphic representation V. Let $\mathcal{H}(V)^+ \subset \mathcal{H}(V)$ be the set of nonnegative Hermitian forms.

Definition. A positive functional on the space $\mathcal{H}(V)$ is an additive map $\rho : \mathcal{H}(V)^+ \to \mathbb{R}^+ \bigcup \infty$.

The basic example of such functionals is the functional $\rho_v(H) = H(v)$ defined for any vector $v \in V$.

We construct below a special family of positive functionals ρ_T parameterized by the real parameter $T \geq 1$ in order to bound coefficients $a_{n,\lambda}$.

Proposition. *There exist a constant C such that for any $T \geq 1$ and an automorphic representation $V \simeq V_\lambda$ we can find a positive functional ρ_T on $\mathcal{H}(V)$ satisfying*

$$\rho_T(H_{\mathcal{O}}^V) \leq CT \tag{25}$$

$$\rho_T(Q_{n,\lambda}^{mod}) \geq 1 \text{ for any } |n|, |\lambda| \leq T \tag{26}$$

Proof. We construct the positive functional ρ_T by integrating an elementary positive functional ρ_{v_T} with the specially chosen vector $v_T \in V_\lambda$ against a smooth compactly supported non-negative function $h \in H(G)$. Namely, let δ be a smooth non-negative function with the support $supp(\delta) \subset [-0.1, 0.1]$ and satisfying $\int \delta(x) dx = 1$. For any real T consider the function $v_T(x) = T \cdot \delta(T(x - 1))$ and view it as a vector in V_λ. We have

$$P_X(v_T) = \|v_T\|_{V_\lambda}^2 = c_1 T \ . \tag{27}$$

We also have

$$Q_{n,\lambda}^{mod}(v_T) \geq c_2 \tag{28}$$

for $\max(|n|, |\lambda|) \leq T$ since v_T has the support on the interval of the size smaller than T around 1 and the kernel of $Q_{n,\lambda}^{mod}$ is given by the oscillating function with the phase having the variation on the support of v_T smaller than $\frac{1}{2}$.

Let $U \subset G$ be a small fixed neighborhood of the identity such that $g^{-1} \cdot [0.9, 1.1] \subset [\frac{1}{2}, \frac{3}{2}]$ for any $g \in U$ under the standard action of G on $\mathbb{R}$. For any $g \in U$ the function $\pi_\lambda(g)v_T$ is very similar to the original function v_T and supported off the singularities of the kernel of $Q_{n,\lambda}^{mod}$. In particular, $\pi_\lambda(g)v_T$ satisfies conditions (27) and (28), possibly with other constants. Hence the positive functional $\Pi(h)\rho_{v_T}$ satisfies the same conditions for any $h \in H(V)$ with $supp(h) \in U$. Taking appropriate h we see that $\rho_T = \Pi(h)\rho_{v_T}$ satisfies the condition (26). Moreover, taking into account Lemma 3.2.1 we see that the condition (25) is also satisfied. $\square$

3.4.2. *Proof of Theorem 3.4.* Taking the positive functional ρ_T from Proposition 3.4.1 we obtain:

$$(29) \qquad CT \geq \rho_T(H_{\mathcal{O}}^V) = \sum_n |a_{n,\lambda}|^2 \rho_T(Q_{n,\lambda}^{mod}) \geq \sum_{|n| \leq T} |a_{n,\lambda}|^2 \rho_T(Q_{n,\lambda}^{mod}) \geq \frac{1}{2} \sum_{|n| \leq T} |a_{n,\lambda}|^2$$

for any $T \geq |\lambda|$.

$\square$

3.4.3. *A conjecture.* While the bound in Theorem 3.4 is sharp, for a single coefficient in the sum it gives only $|a_{n,\lambda}| \ll \max(|n|^{\frac{1}{2}}, |\lambda|^{\frac{1}{2}})$. Such a bound follows easily from the bound on the supremum norm on automorphic representations obtained in [BR2]. This situation is reminiscent of a sharp bound on the average size of the standard Fourier coefficients of Maass forms versus bounds on a single Fourier coefficient towards Ramanujan-Peterson conjecture. On the basis of this analogy we propose the following

Conjecture. *For a fixed closed orbit $\mathcal{O}$ the coefficients $a_{n,\lambda}$ satisfy the following bound*

$$(30) \qquad\qquad |a_{n,\lambda}| \ll (\max(|n|, |\lambda|))^\varepsilon$$

for any $\varepsilon > 0$.

4. RESTRICTION TO CLOSED GEODESICS

In this section we use our main technical result Theorem 3.4 in order to prove Theorem A from the Introduction.

4.1. **Spectral density of a K-fixed vector.** In order to use the Plancherel relation (17) and the average bound (24) we need to compute spectral density of a K-fixed vector $e_0^\lambda \in V_\lambda$ with respect to the model forms $Q_{n,\lambda}^{mod}$. We have

$$(31) \qquad b_{n,\lambda} = d_{n,\lambda}^{mod}(e_0^\lambda) = \int |x|^{-\frac{1}{2} - \lambda/2 + inq}(1+x^2)^{\lambda/2 - \frac{1}{2}} dx = \frac{\Gamma(\frac{1-\lambda+inq}{4})\Gamma(\frac{1-\lambda-inq}{4})}{\Gamma(\frac{1-\lambda}{2})} \ .$$

Where we have used the table integral ([Ma]):

$$(32) \qquad \int |x|^s (1+x^2)^t dx = B\left(\frac{s+1}{2}, -t - \frac{s+1}{2}\right) = \frac{\Gamma(\frac{s+1}{2})\Gamma(-t - \frac{s+1}{2})}{\Gamma(-t)} \ .$$

From the exact expression in (31) and the Stirling formula for the asymptotic of the Γ-function we see that there are constants c_i such that

$$(33) \qquad\qquad |b_{n,\lambda}|^2 \leq \begin{cases} c_1|\lambda|^{-1} & \text{for } |nq| \leq 0.9|\lambda|; \\ c_2|\lambda|^{-\frac{1}{2}}, & \text{for } 0.9|\lambda| \leq |nq| \leq 1.1|\lambda|; \\ c_3 e^{-0.1nq}, & 1.1|\lambda| \leq |nq|. \end{cases}$$

Remark. The constants 0.9 and 1.1 could be substituted by $1 - \sigma$ and $1 + \sigma$ for any $0 < \sigma < 1$.

4.2. **Proof of Theorem A.** Let ϕ_i be a norm one Maass form in an automorphic representation $V \simeq V_\lambda$. We deduce from (17), (24) and (33) that

$$(34) \qquad \frac{1}{\text{length}(\ell)} p^\ell(\phi_i) = \int_{\mathcal{O}} |\phi_i|^2 d\mathcal{O} = H_{\mathcal{O}}^V(e_0^\lambda) = \sum_n |a_{n,\lambda}|^2 Q_n^{mod}(e_0^\lambda) =$$

$$(35) \quad = \sum_{|n| \leq 1.1|\lambda|} |a_{n,\lambda}|^2 |b_{n,\lambda}|^2 + \sum_{1.1|\lambda| \leq |n|} |a_{n,\lambda}|^2 |b_{n,\lambda}|^2 \leq |\lambda|^{-\frac{1}{2}} \sum_{1.1|\lambda| \leq |n|} |a_{n,\lambda}|^2 + C' \leq C|\lambda|^{\frac{1}{2}}$$

since (24) implies that $\sum_{|n| \leq 1.1|\lambda|} |a_{n,\lambda}|^2 \leq C|\lambda|$ and the summation by parts implies that $\sum_{1.1|\lambda| \leq |n|} |a_{n,\lambda}|^2 e^{-0.1qn} \leq C'$. This gives the bound (11) since $|\lambda| \approx \mu^{\frac{1}{2}}$.

$\square$

5. Restriction to geodesic circles

In this section we prove Theorem B on restriction to geodesic circles. The proof goes along same lines as the proof of Theorem A for closed geodesics.

5.1. **Geodesic circles.** We fix a maximal compact subgroup $K \subset G$ and the identification $G/K \to \mathbb{H}$, $g \mapsto g \cdot i$. Let $y \in Y$ be a point and $\pi : \mathbb{H} \to \Gamma \backslash \mathbb{H} \simeq Y$ the projection as before. Let $R_y > 0$ be the injectivity radius of Y at y. For any $r \leq R_y$ we define the geodesic circle of radius r centered at y to be the set $\sigma(r, y) = \{y' \in Y | d(y', y) = r\}$. Since π is a local isometry we have that $\pi(\sigma_{\mathbb{H}}(r, z)) = \sigma(r, y)$ for any $z \in \mathbb{H}$ such that $\pi(z) = y$ where $\sigma_{\mathbb{H}}(r, z)$ is a corresponding geodesic circle in $\mathbb{H}$ (all geodesic circles in $\mathbb{H}$ are Euclidian circles though with the different from y center). We associate to any such circle on Y an orbit of a compact subgroup in X. Namely, any geodesic circle on $\mathbb{H}$ is of the form $\sigma_{\mathbb{H}}(r, z) = hKg \cdot i$ with h, $g \in G$ such that $h \cdot i = z$ and $hg \cdot i \in \sigma_{\mathbb{H}}(r, z)$ (i.e. an h-translation of a standard geodesic circle around $i \in \mathbb{H}$ passing through $g \cdot i$). Note, that the radius of the circle is given by the distance $d(i, g \cdot i)$ and hence $g \notin K$ for a nontrivial circle. Given the geodesic circle $\sigma(r, y) \subset Y$ which gives rise to a circle $\sigma_{\mathbb{H}}(r, z) \subset \mathbb{H}$ and the corresponding elements g, $h \in G$ we consider the compact subgroup $K_\sigma = g^{-1}Kg$ and the orbit $\mathcal{O}_\sigma = hg \cdot K_\sigma \subset X$. Clearly we have $\pi(\mathcal{O}_\sigma) = \sigma$. We endow the orbit $\mathcal{O}_\sigma$ with the unique K_σ-invariant measure $d\mathcal{O}_\sigma$ of the total mass one (from geometric point of view a more natural measure would be the length of σ). We have then $p^\sigma(\phi) = length(\sigma) \int_{\mathcal{O}_\sigma} |\phi|^2 d\mathcal{O}_\sigma$.

We also note that for what follows the restriction $r < R_y$ is not essential. From now on we assume that $\mathcal{O} \subset X$ is any orbit of K'. The restriction $r < R_y$ implies that the projection $\pi(\mathcal{O}) \subset Y$ is a smooth non-self intersecting curve on Y and is not essential for our method.

5.2. **Hermitian forms.** Let σ, $K' = K_\sigma$ and $\mathcal{O} = \mathcal{O}_\sigma$ be as above. We define the non-negative Hermitian form on $C^\infty(X)$ by

$$(36) \qquad H_{\mathcal{O}}(f, g) = \int_{\mathcal{O}} f(o)\bar{g}(o)d\mathcal{O}$$

for any f, $g \in C^\infty(X)$. We will use the shorthand notation $H_{\mathcal{O}}(f) = H_{\mathcal{O}}(f, f)$. Examining the proof of Lemma 3.2.1 we have, in the notations of 3.2.1,

Lemma.   *For any $h \in H(G)$, $h \geq 0$ there exists a constant $C = C_h$ such that*

$$\Pi(h)H_{\mathcal{O}} \leq CP_X \ .$$

$\square$

5.2.1. *Characters.* We fix a point $\dot{o} \in \mathcal{O}$. To a character $\chi : K' \to S^1$ we associate a function $\chi.(\dot{o}k') = \chi(k')$, $k' \in K'$ on the orbit $\mathcal{O}$ and the corresponding functional on $C^\infty(X)$ given by

$$(37) \qquad d^{aut}_{\chi,\mathcal{O}}(f) = \int_{\mathcal{O}} f(o)\bar{\chi}.(o)d\mathcal{O}$$

for any $f \in C^\infty(X)$. The functional $d^{aut}_{\chi,\mathcal{O}}$ is χ-equivariant: $d^{aut}_{\chi,\mathcal{O}}(R(k')f) = \chi(k')d^{aut}_{\chi,\mathcal{O}}(f)$ for any $k' \in K'$, where R is the right action of G on the space of functions on X. For a given orbit $\mathcal{O}$ and the choice of a generator χ_1 of the cyclic group $\hat{K}'$ we will use the shorthand notation $d^{aut}_n = d^{aut}_{\chi_n,\mathcal{O}}$, where $\chi_n = \chi_1^n$. The functions $(\chi_n).$ form an orthonormal basis for the space $L^2(\mathcal{O}, d\mathcal{O})$.

Let V be an irreducible automorphic representation. We introduce non-negative Hermitian forms of rank one $Q^{aut}_n(\cdot) = |d^{aut}_n(\cdot)|^2$ restricted to V and consider the Plancherel formula restricted to V:

$$(38) \qquad H^V_{\mathcal{O}} = \sum_n Q^{aut}_n \ .$$

Let $V \simeq V_\lambda$ be a representation of the principal series. We have $\dim \mathrm{Hom}_{K'}(V_\lambda, \chi) \leq 1$ for any character χ of K' (i.e. the space of K-types is at most one dimensional). This is well-known in representation theory of $PGL_2(\mathbb{R})$ and could be seen from the isomorphism $V_\lambda \simeq C^\infty_{even/odd}(S^1)$, for example. In fact $\dim \mathrm{Hom}_K(V_\lambda, \chi_n) = 1$ iff n is even for V corresponding to a Maass form.

Consider the model $V_\lambda \simeq C^\infty(S^1)$ and the standard vectors (exponents) $e_n = \exp(2\pi i n) \in C^\infty(S^1)$ which form the basis of K-types for the *standard* maximal compact subgroup K. For any n such that $\dim \mathrm{Hom}_K(V_\lambda, \chi_n) = 1$ the function $e'_n = \pi_\lambda(g^{-1})e_n$ defines the model functional on V_λ through $d^{mod}_{n,\lambda}(v) = d^{mod}_{\chi_n,\lambda}(v) = \langle v, e'_n \rangle$ which is χ_n-equivariant

with respect to K'. Introducing the Hermitian forms $Q_n^{mod} = |d_{n,\lambda}(\cdot)|^2$ we arrive at basic relations

$$(39) \qquad Q_{n,\lambda}^{aut} = |a_{n,\lambda}|^2 Q_{n,\lambda}^{mod} \ .$$

$$(40) \qquad H_{\mathcal{O}}^V = \sum_n |a_{n,\lambda}|^2 Q_{n,\lambda}^{mod} \ .$$

5.2.2. *Average bound.* By examining the proof of Theorem 3.4 we arrive at a similar statement.

Theorem. *For a given orbit $\mathcal{O}$ as above there exists a constant $C = C_{\mathcal{O}}$ such that for any $T \geq 1$ the following bound holds*

$$(41) \qquad \sum_{|n| \leq T} |a_{n,\lambda}|^2 \leq C \cdot \max(T, |\lambda|) \ .$$

for any automorphic representation which is isomorphic to a representation of the principal series V_λ.

The proof again is based on the existence of appropriate test functionals on the space of Hermitian forms on V_λ satisfying the same conditions as in Proposition 3.4.1.

$\square$

5.3. **Spectral density.** We are left to compute spectral decomposition of the K-fixed vector $e_0 \in V_\lambda$ with respect to the basis of K'-types. Namely, we need to estimate coefficients

$$(42) \qquad c_{n,\lambda} = d_{n,\lambda}^{mod}(e_0) = < e_0, e'_n > = < e_0, \pi_\lambda(g^{-1})e_n > = < \pi_\lambda(g)e_0, e_n > \ .$$

These are finite K-types matrix coefficients of the spherical vector $e_0 \in V_\lambda$. These matrix coefficients satisfy the following crude bound (compare (33)).

Lemma. *Let $g \in G$ be fixed, $g \neq e$. There exists a constant $c = c_g > 1$ and constants $c_1, \ c_2 > 0$ such that*

$$(43) \qquad |c_{n,\lambda}|^2 \leq \begin{cases} c_1 |\lambda|^{-1} & for \ |2\pi n| \leq 0.9c|\lambda|; \\ c_2 |\lambda|^{-\frac{2}{3}}, & for \ 0.9c|\lambda| \leq |2\pi n| \leq 1.1c|\lambda|; \\ o(n^{-N}) & for \ any \ N > 0, for \ 1.1c|\lambda| \leq |2\pi n|. \end{cases}$$

Proof. We want to estimate quantities $< \pi_\lambda(g)e_0, e_n >$. Namely, we need to estimate the coefficients of the Fourier expansion of $f_\lambda(\theta) = \pi_\lambda(g)e_0(\theta) \in C^\infty(S^1)$. The function f_λ is given by the formula $f_\lambda(\theta) = |g'(\theta)|^{-\frac{1+\lambda}{2}}$. We consider the corresponding oscillatory integral

$$(44) \qquad c_{n,\lambda} = < f_\lambda, e_n > = \int_{S^1} |g'(\theta)|^{-\frac{1}{2}} e^{\frac{1}{2}\lambda \ln |g'(\theta)| - 2\pi in\theta} d\theta \ .$$

Clearly, $g'(\theta)$ is a smooth real valued non-zero function. Let $c = \max_{S^1} d \log |g'|$. It is easy to see (for example by taking the Cartan decomposition $g = k_1 diag(a, a^{-1})k_2$, where k_1, $k_2 \in K$) that the phase of the oscillatory integral (44) has non-degenerate critical points for $|2\pi n| \leq 0.9c|\lambda|$ no critical points for $|2\pi n| \geq 1.1c|\lambda|$ and a critical point with the degeneration of a degree at most 3 (i.e. of a type ct^3 in a local parameter t) for $0.9c|\lambda| \leq |2\pi n| \leq 1.1c|\lambda|$. The bound (43) follows now from the stationary phase method.   □

5.3.1. *Proof of Theorem B.* Examining the proof of Theorem A in Section 4.2 we arrive immediately at the bound (6) in Theorem B.

□

Remark.   The discrepancy in the behavior of coefficients $b_{n,\lambda}$ and $c_{n,\lambda}$ (compare (33) to (43)) is a result of the difference in the type of degeneration of the phase of the oscillatory integrals (31) and (44). This is reflected in the difference of corresponding exponents in bounds in Corollaries A and B. While the phase in the integral in (44) for the coefficients related to geodesic circles has degeneration of a degree at most 3, the corresponding phase in (31) for the coefficients related to closed geodesics has the critical point which coincides with the singularity of the amplitude. We note that underlying average bounds on coefficients $a_{\chi,\lambda}$ relating the geometric Hermitian form $H_\mathcal{O}$ to model Hermitian forms $Q_{\chi,\lambda}^{mod}$ are the same in both cases (compare Theorems 3.4 and 5.2.2).

REFERENCES

[Bo]   Bourgain, J. Geodesic restrictions and L^p-estimates for eigenfunctions of Riemannian surfaces. Linear and complex analysis, 27–35, Amer. Math. Soc. Transl. Ser. 2, 226, Amer. Math. Soc., Providence, RI, 2009.

[BGT1] N. Burq, P. Gérard, N. Tzvetkov, Multilinear eigenfunction estimates for the Laplace spectral projectors on compact manifolds, C. R. Math. Acad. Sci. Paris 338 (2004), no. 5, 359–364, arXiv:math.AP/0310018.

[BGT2] Burq, N.; Grard, P.; Tzvetkov, N. Restrictions of the Laplace-Beltrami eigenfunctions to submanifolds. Duke Math. J. 138 (2007), no. 3, 445–486

[Be]   P. Bérard, On the wave equation on a compact Riemannian manifold without conjugate points, Math. Z. 155 (1977), no. 3, 249–276.

[BR1]   J. Bernstein, A. Reznikov, Analytic continuation of representations, Ann. Math., 150(1999), 329–352.

[BR2]   J. Bernstein, A. Reznikov, Sobolev norms of automorphic functionals, IMRN 2002:40 (2002), 2155-2174.

[BR3]   J. Bernstein, A. Reznikov, Estimates of automorphic functions, Moscow Math. J. 4 (2004), no. 1, 19–37, arXiv:math.RT/0305351.

[He]   S. Helgason, Groups and geometric analysis. Mathematical Surveys and Monographs, 83, AMS, Providence, RI, 2000.

[Du]   W. Duke, Hyperbolic distribution problems and half-integral weight Maass forms, Invent. Math. 92 (1988), no. 1, 73–90.

[GF]   I. Gelfand, S. Fomin, Geodesic flows on manifolds of negative curvature. Uspekhi Mat. Nauk., 7, no. 1 (1952), 118–137.

[G1]   I. Gelfand, G. Shilov , Generalized Functions. vol. 1, Academic Press, 1964.

[G5]   I. Gelfand, M. Graev, N. Vilenkin, Generalized Functions. vol. 5, Academic Press, 1966.
[G6]   I. Gelfand, M. Graev, I. Piatetski-Shapiro, Representation Theory and Automorphic Forms. Saunders, 1969.
[Go]   A. Good, Beiträge zur Theorie der Dirichletreihen, die Spitzenformen zugeordnet sind, J. Number Theory 13 (1981), no. 1, 18–65.
[Iw]   H. Iwaniec, Spectral methods of automorphic forms. Second edition. Graduate Studies in Mathematics, 53. AMS, Providence, 2002.
[ISa]   H. Iwaniec, P. Sarnak, L^∞ norms of eigenfunctions of arithmetic surfaces. Ann. of Math. (2) 141 (1995), no. 2, 301–320.
[L]   S. Lang, $SL_2(\mathbb{R})$. Springer GTM105, 1985.
[KSa]   S. Katok, P. Sarnak, Heegner points, cycles and Maass forms. Israel J. Math. 84 (1993), no. 1-2, 193–227.
[M]   H. Maass, Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, Math. Ann. 121, (1949). 141–183.
[Ma]   W. Magnus et al., Formulas and Theorems for the Special Functions, Springer, 1966.
[MV]   P. Michel, A.Venkatesh, The subconvexity problem for GL_2 to appear in Publ. Math IHES, arXiv:0903.3591
[NPS]   F. Nazarov, L. Polterovich, M. Sodin, Sign and area in nodal geometry of Laplace eigenfunctions, arXiv:math.AP/0402412, preprint, 2004.
[Pe]   Y. Petridis, On squares of eigenfunctions for the hyperbolic plane and a new bound on certain L-series, Internat. Math. Res. Notices, no. 3, (1995), 111–127.
[Pitt] Pitt, Nigel J. E. A sum formula for a pair of closed geodesics on a hyperbolic surface. Duke Math. J. 143 (2008), no. 3, 407–435.
[RS]   Z. Rudnick, P. Sarnak, The behaviour of eigenstates of arithmetic hyperbolic manifolds, Comm. Math. Phys. 161 (1994), no. 1, 195–213.
[Sa1]   P. Sarnak, Arithmetical Quantum Chaos, Israeli Math. Conference Proc., vol. 8, 1995.
[Sa2]   P. Sarnak, Spectra of hyperbolic surfaces. Bull. AMS (N.S.) 40 (2003), no. 4, 441–478.
[Sa3]   P. Sarnak, Integrals of products of eigenfunctions, Internat. Math. Res. Notices, no. 6, (1994), 251–261.
[Se]   A. Selberg, Harmonic analysis and discontinuous groups (1956), in Collected works, Springer-Verlag, New York (1989), 423–463.
[So]   C. D. Sogge, Kakeya-Nikodym averages and L^p-norms of eigenfunctions, preprint, arXiv:0907.4827
[Ta]   D. Tataru, On the regularity of boundary traces for the wave equation. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 26 (1998), no. 1, 185–206.

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